An alternative presentation for the Hilden group

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Let $B_{2n}$ be the braid group on $2n$ strings with the standard generators $\sigma_1, \ldots, \sigma_{2n-1}$. Define the following elements of the braid group.

\[
\begin{align*}
p_i &= \sigma_2 \sigma_{2i-1} \sigma_{2i+1}^{-1} \sigma_{2i}^{-1} & \text{for } i \in \{1, \ldots, n-1\} \\
s_i &= \sigma_2 \sigma_{2i-1} \sigma_{2i+1} \sigma_{2i} & \text{for } i \in \{1, \ldots, n-1\} \\
t_i &= \sigma_{2i-1} & \text{for } i \in \{1, \ldots, n\}
\end{align*}
\]

**Theorem 1** (Brendle–Hatcher[1], Tawn[2]). The group Hilden (or Wicket) group $H_{2n}$ has a presentation with generators $p_i$, $s_i$ and $t_k$ for $1 \leq i, j < n$ and $1 \leq k \leq n$ and the following relations.

\[
\begin{align*}
p_ip_j &= p_jp_i & \text{for } |i - j| > 1 & (P1) \\
p_ip_jp_i &= p_jp_ip_j & \text{for } |i - j| = 1 & (P2) \\
s_is_j &= s_is_j & \text{for } |i - j| > 1 & (P3) \\
s_is_i &= s_is_is_j & \text{for } |i - j| = 1 & (P4) \\
p_is_j &= s_js_i & \text{for } |i - j| > 1 & (P5) \\
p_is_{i+1}s_i &= s_is_{i+1}p_{i+1} & (P6) \\
p_{i+1}ps_{i+1} &= s_ip_{i+1}p_i & (P7) \\
p_{i+1}s_is_{i+1} &= s_is_{i+1}p_i & (P8) \\
p_is_{i+p} &= s_is_{i+p} & (P9) \\
p_is_j &= t_jp_i & \text{for } j \neq i, \text{ or } i + 1 & (P10) \\
p_is_{i+1} &= t_ip_i & (P11) \\
s_is_j &= t_jp_i & \text{if } j \neq i \text{ or } i + 1 & (P12) \\
s_is_j &= t_jp_i & \text{if } \{i, i + 1\} = \{j, k\} & (P13) \\
t_is_{i+j} &= t_jl_i & \text{for } 1 \leq i, j \leq n & (P14)
\end{align*}
\]

Let $q_i = \sigma_2 \sigma_{2i-1} \sigma_{2i+1} \sigma_{2i}$ for $i \in \{1, \ldots, n-1\}$. Then $q_i = p_i s_i$ and we have the following.

**Theorem 2.** The group $H_{2n}$ has a presentation with generators $q_i$, $s_i$ and $t_j$ for
$1 \leq i < n \text{ and } 1 \leq j \leq n$ and the following relations.

\begin{align*}
q_i q_j &= q_j q_i \quad \text{for } |i - j| > 1 \quad (Q1) \\
s_i s_j &= s_j s_i \quad \text{for } |i - j| > 1 \quad (Q2) \\
t_i t_j &= t_j t_i \quad (Q3) \\
q_i s_j &= s_j q_i \quad \text{for } |i - j| > 1 \quad (Q4) \\
q_i t_j &= t_j q_i \quad \text{for } j \neq i + 1 \quad (Q5) \\
s_i t_j &= t_j s_i \quad \text{if } j \neq i \text{ or } i + 1 \quad (Q6) \\
s_i t_j &= t_k s_i \quad \text{if } \{i, i + 1\} = \{j, k\} \quad (Q7) \\
s_i s_j s_i &= s_j s_i s_j \quad \text{for } |i - j| = 1 \quad (Q8) \\
q_i s_j &= s_j s_i q_j \quad \text{for } |i - j| = 1 \quad (Q9) \\
q_{i+1} s_i q_{i+1} s_i &= s_i q_{i+1} s_i q_{i+1} \quad (Q10) \\
q_i t_{i+1} q_i &= s_i t_i s_i \quad (Q11) \\
s_i q_i q_{i+1} s_i q_{i+1} &= q_{i+1} s_i q_{i+1} q_i s_i \quad (Q12) \\
\end{align*}

**Proof.** By substituting $p_i = q_i s_i^{-1}$ we see that (P1)–(P14) are equivalent to the following.

\begin{align*}
q_i s_i^{-1} q_j s_j^{-1} &= q_j s_j^{-1} q_i s_i^{-1} \quad \text{for } |i - j| > 1 \quad (1) \\
q_i s_i^{-1} q_j s_j^{-1} q_i s_i^{-1} &= q_j s_j^{-1} q_i s_i^{-1} q_j s_j^{-1} \quad \text{for } |i - j| = 1 \quad (2) \\
s_i s_j &= s_j s_i \quad \text{for } |i - j| > 1 \quad (3) \\
s_i s_j &= s_j s_i \quad \text{for } |i - j| = 1 \quad (4) \\
q_i s_i^{-1} s_j &= s_j q_i s_i^{-1} \quad \text{for } |i - j| = 1 \quad (5) \\
q_i s_i^{-1} s_{i+1} &= s_{i+1} s_i q_{i+1} s_i^{-1} \quad (6) \\
q_{i+1} s_{i+1} q_i s_i^{-1} &= s_{i+1} s_i q_{i+1} s_i^{-1} \quad (7) \\
q_{i+1} s_{i+1} q_i s_i^{-1} &= s_{i+1} q_i s_{i+1} q_i s_i^{-1} \quad (8) \\
q_i s_i^{-1} t_i q_i s_i^{-1} &= s_i t_i \quad (9) \\
q_i s_i^{-1} t_j &= t_j q_i s_i^{-1} \quad \text{for } j \neq i, \text{ or } i + 1 \quad (10) \\
q_i s_i^{-1} t_{i+1} &= t_i q_i s_i^{-1} \quad (11) \\
s_i t_j &= t_j s_i \quad \text{if } j \neq i \text{ or } i + 1 \quad (12) \\
s_i t_j &= t_k s_i \quad \text{if } \{i, i + 1\} = \{j, k\} \quad (13) \\
t_i t_j &= t_j t_i \quad \text{for } 1 \leq i, j \leq n \quad (14) \\
\end{align*}

Now (3), (4), (12), (13) and (14) are the same as the corresponding $Q$ relations. Modulo (Q4) relation (1) is equivalent to (Q1). Modulo (Q2) relation (5) is equivalent to (Q4). Modulo (Q8) relations (6) and (8) are equivalent to (Q9). Modulo (Q7) relation (9) is equivalent to (Q11). Modulo (Q6) relation (10) is equivalent to (Q5). Modulo (Q7) relation (11) is equivalent to (Q5).

This only leaves (2) and (7). Relation (7) follows by the following.

\begin{align*}
q_{i+1} s_{i+1} q_i s_i^{-1} s_{i+1} &= (Q8) \\
\end{align*}
In the lefthand side is of the same length as the word on the righthand side. So if we have the following.

\[
\begin{align*}
  q_i s_i^{-1} q_i s_i^{-1} q_i s_i^{-1} & \quad \text{(Q9)} \\
  = q_i s_i^{-1} q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q8)} \\
  = q_i s_i^{-1} q_i q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q10)} \\
  = q_i q_i s_i q_i q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q12)} \\
  = s_i^{-1} q_i q_i s_i q_i q_i s_i q_i s_i^{-1} q_i q_i s_i^{-1} & \quad \text{(Q9)} \\
  = s_i^{-1} q_i q_i s_i q_i q_i s_i q_i s_i^{-1} q_i q_i s_i^{-1} & \quad \text{(Q10)} \\
  = q_i q_i s_i q_i q_i s_i q_i q_i s_i q_i s_i^{-1} & \quad \text{(Q9)} \\
  = q_i q_i s_i q_i q_i s_i q_i q_i s_i q_i s_i^{-1} & \quad \text{(Q9)} \\
\end{align*}
\]

Relation (2) follows by the following.

\[
\begin{align*}
  q_i s_i^{-1} q_i s_i^{-1} q_i s_i^{-1} & \quad \text{(Q9)} \\
  = q_i s_i^{-1} q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q8)} \\
  = q_i q_i s_i q_i q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q10)} \\
  = q_i q_i s_i q_i q_i s_i q_i s_i^{-1} s_i^{-1} s_i^{-1} & \quad \text{(Q12)} \\
  = s_i^{-1} q_i q_i s_i q_i q_i s_i q_i s_i^{-1} q_i q_i s_i^{-1} & \quad \text{(Q9)} \\
  = s_i^{-1} q_i q_i s_i q_i q_i s_i q_i s_i^{-1} q_i q_i s_i^{-1} & \quad \text{(Q10)} \\
  = q_i q_i s_i q_i q_i s_i q_i q_i s_i q_i s_i^{-1} & \quad \text{(Q9)} \\
  = q_i q_i s_i q_i q_i s_i q_i q_i s_i q_i s_i^{-1} & \quad \text{(Q9)} \\
\end{align*}
\]

\[\square\]

**Question.** Is (Q12) really necessary?

The new presentation has the property that for each relation the word on the lefthand side is of the same length as the word on the righthand side. So if we let \( \epsilon_{\mathbb{H}_2}(x) \) be the sum of the exponents in a word representing \( x \) then this is well defined. For the positive Hilden monoid \( \mathbb{H}_2^+ = \langle q_i, s_i, t_j \rangle^+ \) the length of a word is equal to the exponent sum and so we have the following corollary.

**Corollary 3.** The positive paths in the Hilden group are undistorted in the braid group in the sense that for any \( x \in \mathbb{H}_2^+ \) we have the following.

\[
\ell_{\mathbb{H}_2}(x) \leq \ell_{\mathbb{B}_2}(x) \leq 4 \ell_{\mathbb{H}_2}(x)
\]

**Note.** In \( \mathbb{B}_6 \) we have the following.

\[
\Delta = s_1 s_2 s_1 t_1 t_2 t_3
\]

\[
\Delta^2 q_i^{-1} = t_3 s_2 s_1 t_2 t_3 q_2 s_1 t_3 s_2 t_3
\]

\[
\Delta^2 q_i^{-1} = t_2 s_1 t_3 s_2 t_3 q_2 s_1 t_3 s_2 t_3
\]

\[
\Delta s_i^{-1} = t_2 s_1 t_3 s_2 t_3
\]

\[
\Delta s_i^{-1} = t_1 t_3 s_2 t_3
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\[
\Delta t_i^{-1} = s_2 s_1 t_3 s_2 t_3
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\[
\Delta t_i^{-1} = t_3 s_2 s_1 t_3 s_2 t_3
\]

\[
\Delta t_i^{-1} = t_3 s_2 s_1 t_3 s_2 t_3
\]

This suggests the following conjecture.

**Conjecture 4.** For every \( h \in \mathbb{H}_2 \), there exists \( k \in \mathbb{N} \) such that \( \Delta^k h \in \langle q_i, s_i, t_j \rangle^+ \).

**Question.** Does \( \mathbb{H}_2 \cap \mathbb{B}_6^+ = \langle q_i, s_i, t_j \rangle^+ \)?

No.

**Conjecture 5.** \( \mathbb{H}_2 \cap \mathbb{B}_6^+ \) is not finitely generates.
References
